Math 250 3.2 Working with the Derivative

Objectives

1) Given the graph of the function, graph the derivative.
    a. For a given value of $x = a$, identify the point $(a, f(a))$ on the graph.
    b. Draw or imagine the tangent line to the graph of f at $(a, f(a))$.
    c. Estimate slope f the tangent line by calculating rise and run. The resulting number is $f'(a)$.
    d. Plot the point $(a, f'(a))$ on the graph of the derivative f'.
    e. Repeat for other values of $x = a$ until a thorough image of f' is obtained.
2) Given the graph of the derivative, infer information about the function.
    a. Given the slope information using the same x-axis, we can infer the increasing and decreasing behavior of the original function, but NOT its exact location.
3) Identify points where a graph is defined not differentiable.
    a. Corner point – continuous but not differentiable because $L \neq R$ in the derivative
    b. Cusp – continuous but not differentiable because derivative is unbounded (and likely $L \neq R$)
    c. Vertical tangent - continuous but not differentiable because derivative is unbounded
4) Understand relationship between differentiability and continuity using logic.
    a. A differentiable function must be continuous.
    b. But a continuous function might not be differentiable.
    c. If a function is not continuous, it is not differentiable.
    d. A differentiable function is also called a "smooth" function, because it is continuous and is without cusps or corners.

Review of Basic Statements in Logic

- When a statement has the form "if p then q", p is the "<u>hypothesis</u>" and q is the "<u>conclusion</u>".
- In a <u>true statement</u>, the connection between p and q must always be true: Every time p is true, q is true.
- It takes only one example where p is true but q is false (called a <u>counterexample</u>) to prove the statement is logically false. (If a statement seems to be "sometimes true", it is false in logic.)
- The <u>contrapositive</u> of the statement "if p then q" is constructed as "if not q then not p", and is logically equivalent to the original statement. (Both the statement and the contrapositive are true, or both false.)

Understand relationship between differentiability and continuity

A function f is called <u>differentiable</u> at $(c, f(c))$ if the $\lim\limits_{\Delta x \to 0} \dfrac{f(c+\Delta x) - f(c)}{\Delta x}$ (two-sided) exists.

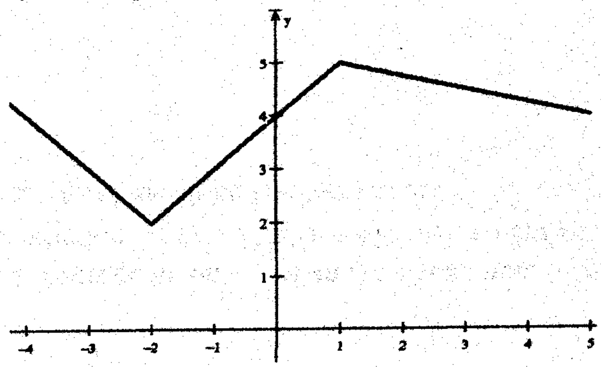
A function f is called <u>differentiable</u> on (a, b) if the $\lim\limits_{\Delta x \to 0} \dfrac{f(c+\Delta x) - f(c)}{\Delta x}$ (two-sided) exists for all c in (a, b)

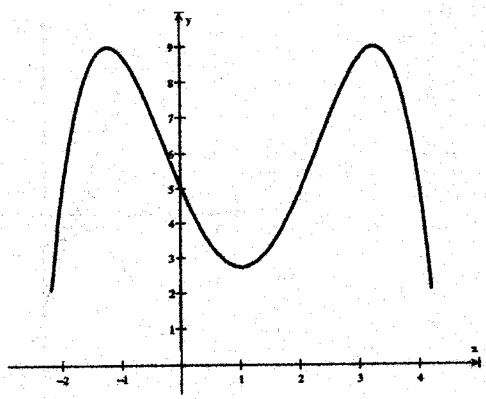
If this limit does not exist, the function f is <u>nondifferentiable</u> at $(c, f(c))$, meaning any of the following:

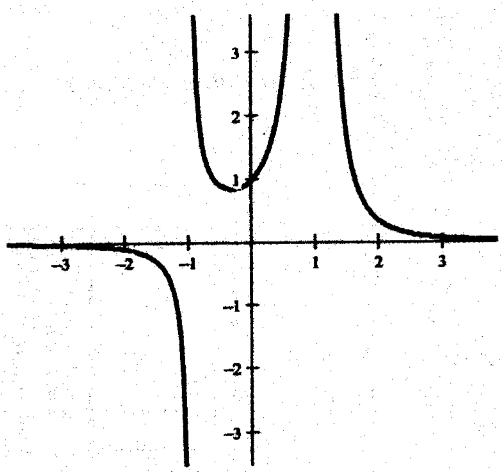
1) The function is not defined ($f(c)$ undefined), so there's no tangent line,
2) The slope of the tangent line from the left is not equal to the slope of the tangent line from the right,
3) The slope of the tangent line at $(c, f(c))$ is infinite – it's a vertical tangent line,
4) The slope of the tangent line oscillates.

Examples and Practice

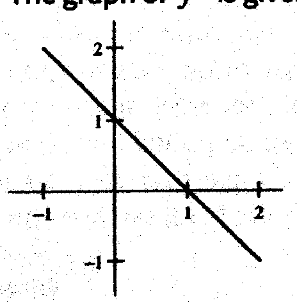
1) The graph of f is given. Sketch the graph of f'.

a. 

b. 

c. 

2) The graph of f' is given. Sketch a graph of f.



3) Sketch each graph and determine the values of x where it is not continuous and not differentiable, if any.

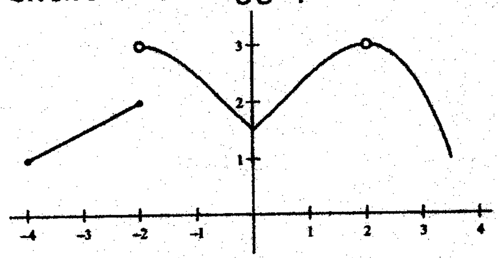
a. $f(x)=\begin{cases} x+2 & x>0 \\ x-2 & x\le 0 \end{cases}$

b. $f(x)=|x|$

c. $f(x)=\sqrt[3]{x}$

d. $f(x)=\dfrac{(x^2-1)}{(x-1)}$

4) Given the following graph of f:



a. Find the values of x in (-4, 3.5) at which f is not continuous.
b. Find the values of x in (-4, 3.5) at which f is not differentiable.
c. Sketch the graph of the derivative of f.

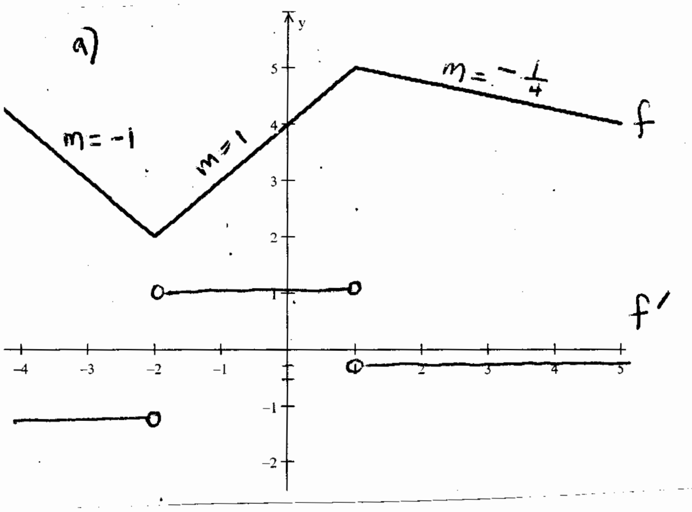
5) Consider whether the following statements are true or false using examples above or others you choose.
a. If f is differentiable, then f is continuous.
b. If f is NOT continuous, then f is NOT differentiable.
c. If if is continuous, then f is differentiable.

6) Prove Theorem 3.1: If *f* is differentiable at x=*a*, then *f* is continuous at x=*a*.

Math 250 3.2

(1) The graph of f is given. Sketch the graph of f'

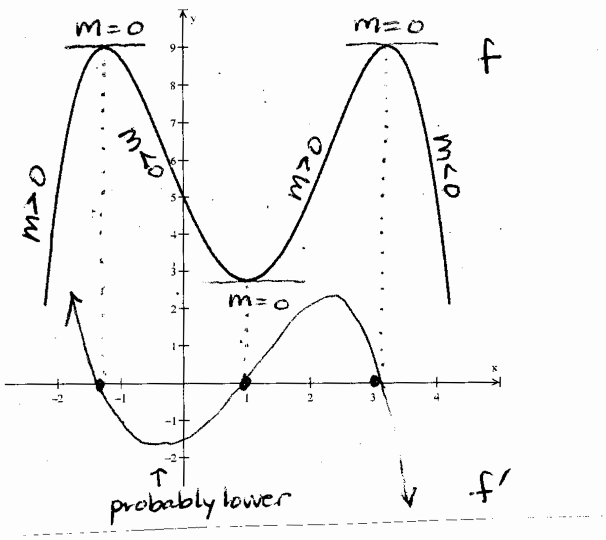
a)



For any value of x along a linear segment, the tangent line is the same as the function.

- Estimate the slope of each linear segment

b)



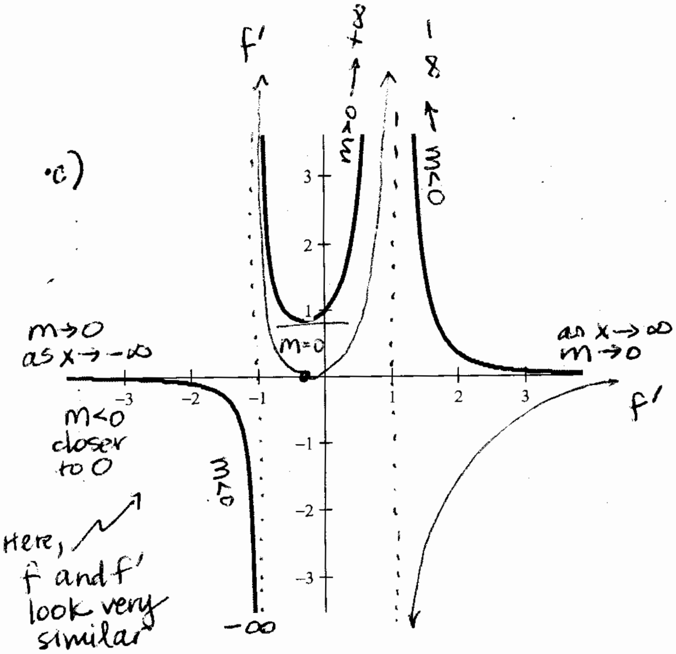
Begin at values of x for which the tangent line is horizontal → horizontal is $m=0$.

Notice if tangent lines are uphill / having positive slope

or downhill \ having negative slope.

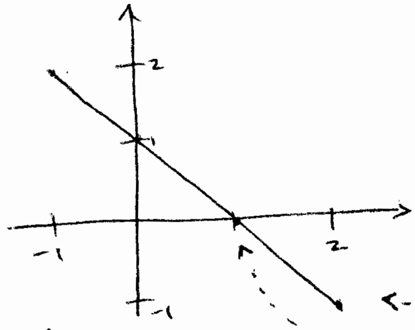
Negative values of the derivative f' are plotted as negative y-coordinates (below the x-axis) on graph of f'.

c)

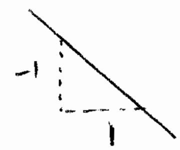


Math 250

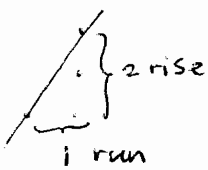
(2) the graph of f' is



when $x = 2$
$f'(2) = -1$
m_{tan} @ $x = 2$



when $x = -1$
$f'(-1) = 2$
m_{tan} @ $x = -1$



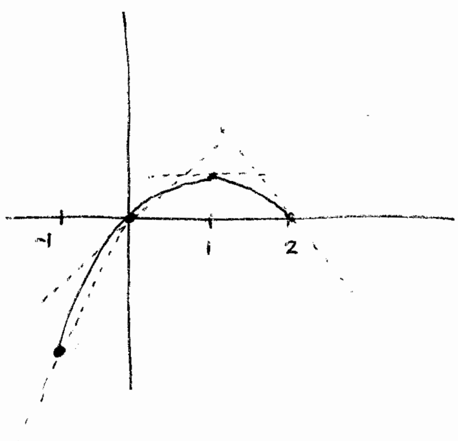
when $x = 0$
$f'(0) = 1$
m_{tan} @ $x = 0$

when $x = 1$
$f'(1) = 0$
m_{tan} @ $x = 1$

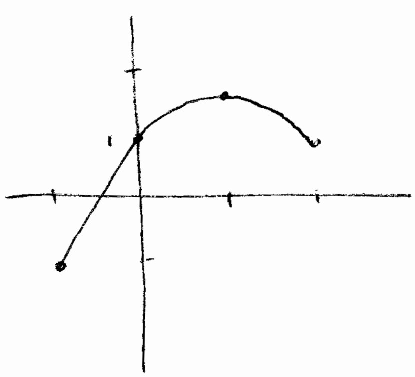
horizontal

But we do not have enough information → we know slopes, not y-coordinates!

Let's just choose a y-coord: at $x=0$, $y=0$ $(0,0)$
with tangent line having $m=1$
the graph must go uphill at $x=0$.



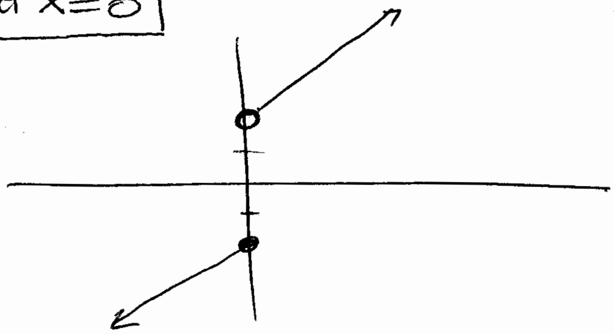
If we had chosen a different y-coord: at $x=0$, $y=1$ $(0,1)$
we translate this same shape up one unit.



M250 3.2

3
a) Consider $f(x) = \begin{cases} x+2 & x > 0 \\ x-2 & x \leq 0 \end{cases}$

Not continuous at $x=0$



The slopes of the two pieces are the same, $m=1$.
Why is this not differentiable at $x=0$?

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \text{derivative from the right } f'_+(a)$$

$$= \lim_{x\to 0^+} \frac{x+2-(-2)}{x}$$

$$= \lim_{x\to 0^+} \frac{x+4}{x}$$

$$= \lim_{x\to 0^+} \left(\frac{x}{x} + \frac{4}{x}\right)$$

$$= \lim_{x\to 0^+} \left(1 + \frac{4}{x}\right) = +\infty, \text{ DNE unbounded}$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \text{derivative from the left } f'_-(a)$$

$$= \lim_{x\to 0^-} \frac{x-2-(-2)}{x-0}$$

$$= \lim_{x\to 0^-} \frac{x}{x}$$

$$= \lim_{x\to 0^-} 1$$

$$= 1$$

$+\infty \neq 1$

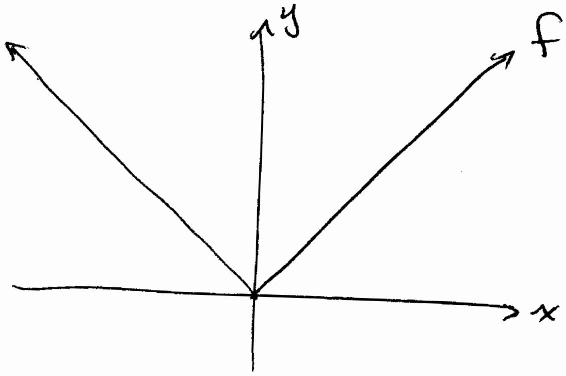
so $f'_+(a) \neq f'_-(a)$ making the 2-sided limit DNE.

$\Rightarrow$ The graph is NOT differentiable at $x=0$

Math 250 3.2

sketch, continuous, differentiable?

b) $f(x) = |x|$



$\boxed{\text{No values of } x \text{ where } f \text{ is discontinuous}}$

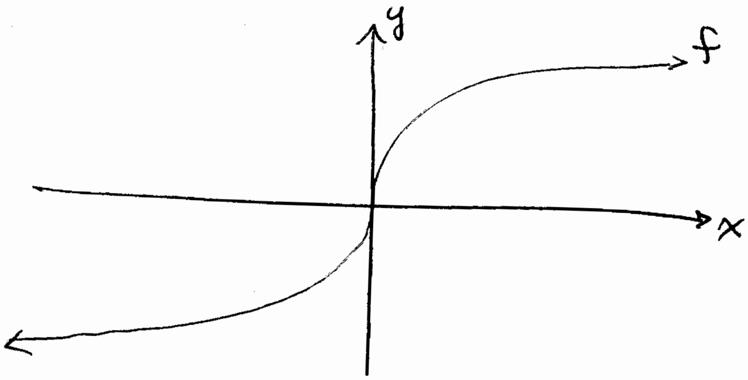
continuous $(-\infty, \infty)$

differentiable $(-\infty, 0) \cup (0, \infty)$

(cusp $L \neq R$ for slope at $x=0$)

$\boxed{\text{Not differentiable at } x=0}$

c) $f(x) = \sqrt[3]{x}$



$\boxed{\text{No values of } x \text{ where } f \text{ is discontinuous}}$

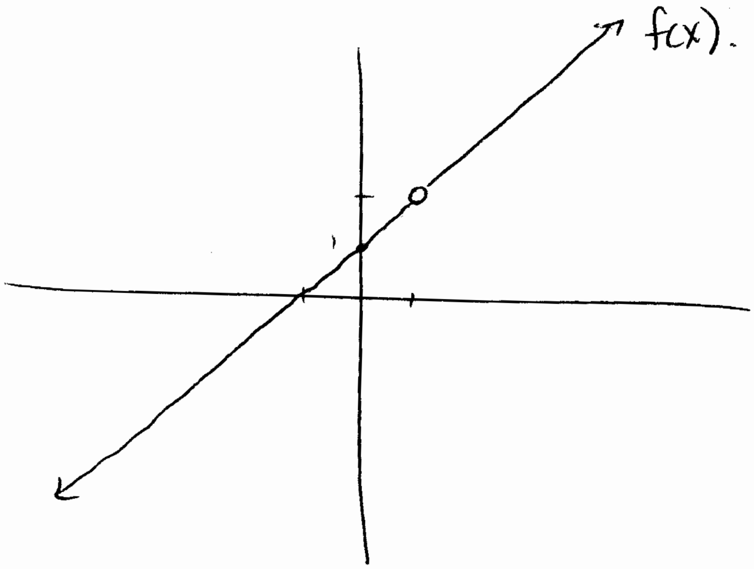
continuous $(-\infty, \infty)$

differentiable $(-\infty, 0) \cup (0, \infty)$

(unbounded slope at $x=0$)

$\boxed{\text{Not differentiable at } x=0}$

d) $f(x) = \dfrac{(x^2-1)}{(x-1)} = \dfrac{(x+1)(x-1)}{(x-1)} = (x+1) \quad x \neq 1$ & undefined at $x=1$.



continuous $(-\infty, 1) \cup (1, \infty)$

differentiable $(-\infty, 1) \cup (1, \infty)$.

$\boxed{\begin{array}{l}\text{Not continuous at } x=1 \\ \text{Not differentiable at } x=1\end{array}}$

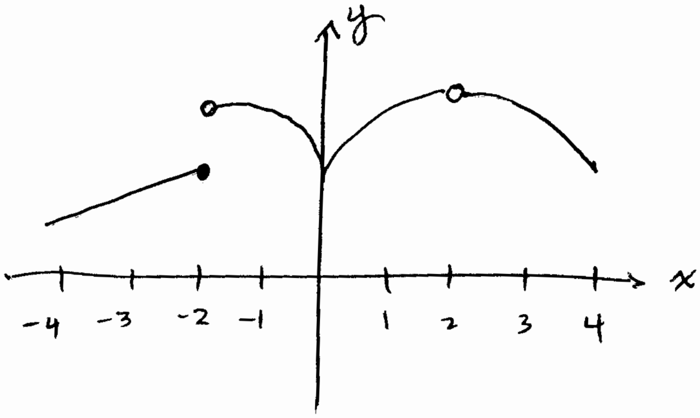
Math 250

(4) Given the graph of f

a) Find the values of x in $(-4, 3.5)$ at which f is not continuous.

b) Find the values of x in $(-4, 3.5)$ at which f is not differentiable.

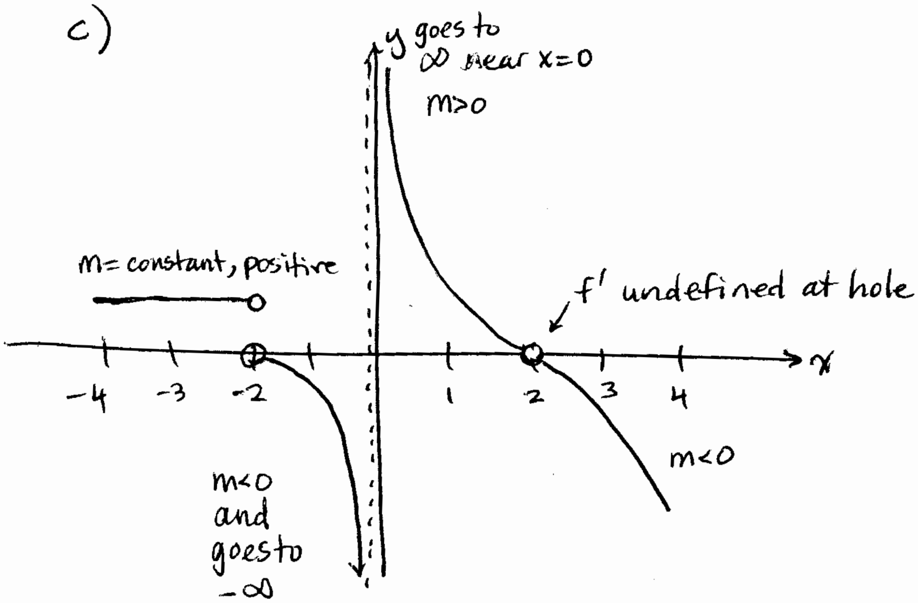
c) Sketch graph of derivative.



a) discontinuous at $x=-2$ (jump) and $x=2$ (hole)

b) nondifferentiable at $x=-2$ (not cont), $x=0$ (vertical tangent, cusp), and $x=2$ (not cont)

c)



Math 250

5

a) If f is differentiable, then f is continuous.

- We have no counterexample, so this seems to be true.

b) If f is NOT continuous, then f is NOT differentiable.

- We have no counterexample, so this seems to be true.

c) If f is continuous, then f is differentiable.

3 b) is continuous but not differentiable at $x=0$.

3 c) " " " " " " " "

4 is continuous but not differentiable at $x=0$.

We have three counterexamples, so this statement is false.

⑥ Theorem 3.1 Differentiable Implies Continuous

If f is differentiable at a, then f is continuous at a.

1) Recall f differentiable means

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \quad \text{exists} \quad (\text{2-sided!})$$

2) Recall f continuous means

$$\lim_{x \to a} f(x) = f(a)$$

(2 sided limit exists <u>and</u> equals function value).

We can use the first statement as a known fact.
We want to conclude the second statement.

<u>Proof</u>:

First, some fancy algebra:

$$f(x) = f(x) - f(a) + f(a).$$

add and subtract the same.

$$f(x) = \frac{(f(x)-f(a))}{x-a} \cdot (x-a) + f(a)$$

mult and divide the same. $x \neq a$ important!

Take the limit of both sides as $x \to a$:

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left[\frac{(f(x)-f(a))}{(x-a)} \cdot (x-a) + f(a) \right]$$

Use limit properties on RHS.

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x \to a} (x-a) + \underbrace{\lim_{x \to a} f(a)}.$$

$$\lim_{x \to a} f(x) = \underbrace{f'(a)}_{\text{defn}} \cdot \underbrace{(a-a)}_{\text{subst}} + f(a)$$

$f(a)$ is constant as $x \to a$

$$= f'(a) \cdot 0 + f(a)$$

$$= 0 + f(a)$$

$$\lim_{x \to a} f(x) = f(a)$$

← but this means f is continuous at a!

QED.